

An inhomogeneous Lax representation for the Hirota equation

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Abstract

Motivated by recent work on quantum integrable models without $U(1)$ symmetry, we show that the $sl(2)$ Hirota equation admits a Lax representation with inhomogeneous terms. The compatibility of the auxiliary linear problem leads to a new consistent family of Hirota-like equations.

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1 Introduction and summary

The Hirota equation is ubiquitous in the theory of quantum integrable systems [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. For an $sl(2)$ -invariant periodic quantum spin chain (see Appendix A for details), the Hirota equation takes the form [10, 11]¹

$$T_k^+ T_k^- - T_{k+1} T_{k-1} = \phi^{[k]} \bar{\phi}^{[-k]}, \quad T_{-1} = 0, \quad k = 0, 1, 2, \dots, \quad (1.1)$$

where $f^\pm = f(u \pm \frac{i}{2})$ and $f^{[\pm k]} = f(u \pm \frac{ik}{2})$ for any function $f(u)$. Here $T_k(u)$ is the transfer matrix constructed with a spin- $k/2$ auxiliary space [2, 13]; in particular, $T_1(u)$ is the fundamental transfer matrix. These transfer matrices mutually commute ($[T_k(u), T_j(v)] = 0$), and obey the Hirota equation (1.1) with

$$\phi(u) = (u + \frac{i}{2})^N, \quad \bar{\phi}(u) = (u - \frac{i}{2})^N. \quad (1.2)$$

The eigenvalues corresponding to simultaneous eigenvectors of these transfer matrices (which we also denote by $T_k(u)$) evidently also obey the Hirota equation, and we henceforth regard $T_k(u)$ as a scalar function.

It is well known that the Hirota equation (1.1) admits a Lax representation through the auxiliary linear problem (see [9, 11] and references therein)

$$T_{k+1} Q^{[k]} - T_k^- Q^{[k+2]} = \phi^{[k]} \bar{Q}^{[-k-2]}, \quad (1.3)$$

$$T_{k-1} \bar{Q}^{[-k-2]} - T_k^- \bar{Q}^{[-k]} = -\bar{\phi}^{[-k]} Q^{[k]}, \quad (1.4)$$

where the function $Q(u)$ has the conjugation property $Q(u)^* = \bar{Q}(u^*)$. However, in order to reproduce the celebrated Baxter T-Q relation, we henceforth restrict our attention to the case of real analytic Q : $Q(u)^* = Q(u^*)$, which implies that

$$Q(u) = \bar{Q}(u) \quad (1.5)$$

for any complex u . Note that the Hirota equation (1.1) with $k = 0$ can be satisfied by setting

$$T_0 = \phi^-, \quad \bar{\phi} = \phi^{[-2]}. \quad (1.6)$$

It then follows from the first Lax equation (1.3) with $k = 0$ (or alternatively from the second Lax equation (1.4) with $k = 1$) that

$$T_1 Q = \bar{\phi} Q^{[2]} + \phi Q^{[-2]}, \quad (1.7)$$

which is the important Baxter T-Q equation [14]. Since (1.3) is linear, it can be solved for all the T_k in terms of Q , and therefore it gives T-Q-like equations for the higher (fused) transfer matrices.

The conventional wisdom has been that (1.3)-(1.4) is the unique Lax representation for the Hirota equation. However, it has recently been shown that quantum integrable models

¹In general, the transfer matrix $T_{a,s}$ can have two subscripts, corresponding to the representation of the auxiliary space given by a rectangular Young tableau with s rows and a columns. For simplicity, we focus here exclusively on the $sl(2)$ case, where T has a single subscript $T_s = T_{1,s}$.

without $U(1)$ symmetry (such as the open XXX spin-1/2 chain with non-diagonal boundary terms, see Appendix B for details) can be solved using a Baxter T-Q equation with an inhomogeneous term, i.e. with the structure [15, 16, 17, 18, 19, 20]

$$T_1 Q = \bar{\phi} Q^{[2]} + \phi Q^{[-2]} + \Delta, \quad (1.8)$$

where $\Delta(u)$ is real analytic (in particular, real for real u) and independent of Q . Indeed, such an inhomogeneous term is necessary in order for the function $Q(u)$ to be a *polynomial* in u . The transfer matrices for such models [21], constructed using non-diagonal boundary S-matrices [22, 23], still obey [24, 25, 26] the Hirota equation, albeit in a slightly modified form,

$$T_k^+ T_k^- - T_{k+1} T_{k-1} = T_{2,k}, \quad T_{-1} = 0, \quad k = 0, 1, 2, \dots, \quad (1.9)$$

where $T_{2,k}$ is given by the quantum determinant (3.2). This naturally raises the question: does the Hirota equation (1.9) admit a Lax representation with inhomogeneous terms?

We answer this question here in the affirmative. Indeed, we show that such a Lax representation is given by (3.13)-(3.14)

$$T_{k+1} Q^{[k]} - \bar{\phi}^{[k]} T_k^- Q^{[k+2]} = X_k Q^{[-k-2]} + \sum_{l=0}^k \psi_{l,k} \Delta^{[2l-k]} T_l^{[l-k-1]}, \quad (1.10)$$

$$\phi^{[-k]} T_{k-1} Q^{[-k-2]} - T_k^- Q^{[-k]} = -Y_k Q^{[k]} - \sum_{l=0}^{k-1} \bar{\psi}_{l,k-1}^- \Delta^{[k-2l-2]} T_l^{[k-l-1]}, \quad (1.11)$$

where X_k, Y_k and $\psi_{l,k}$ are given by (3.7) and (3.15). The key new point is the appearance of terms containing Δ , which do not contain Q and therefore are “inhomogeneous” terms. In particular, (1.10) reduces to (1.8) for $k = 0$. As the equations (1.10) are still linear, they can be solved for all T_k in terms of Q . We remark that equivalent expressions for T_k in terms of Q were obtained earlier by means of a generating function [17]. An AdS/CFT generalization of this generating function was proposed in [27], and it was subsequently used in [28] to compute wrapping corrections.

Interestingly, the compatibility of the system (1.10)-(1.11) leads to a family of Hirota-like equations (3.23)

$$T_{k+1} T_{k-a-1}^{[a]} - T_k^- T_{k-a}^{[a+1]} + T_{2,k-a}^{[a]} T_a^{[a-k-1]} = 0, \quad a = 0, 1, \dots, k-1, \quad (1.12)$$

whose particular case $a = 0$ coincides with the original Hirota equation (1.9). To our knowledge, the bilinear relations (1.12) with $a > 0$ are new. We show that these relations are consistent with the Hirota equation (1.9) by first solving the latter to obtain a determinant expression for T_k in terms of T_1 , and by then judiciously applying Plücker relations.

The outline of this paper is as follows. In Sec. 2 we briefly review for the periodic spin chain how the compatibility of the auxiliary linear problem implies the Hirota equation. In Sec. 3 we turn to the open spin chain. We present both homogeneous and inhomogeneous Lax representations of the Hirota equation. We derive the compatibility conditions for the

auxiliary problem (1.10)-(1.11), and show that they are satisfied if the Hirota-like equations (1.12) are obeyed. In Sec. 3.3 we solve the Hirota equation (1.9) to obtain a determinant expression for T_k in terms of T_1 , see (3.25) and (3.26). In Sec. 3.4 we use Plücker relations to show that this solution is also a solution of the Hirota-like equations. In Sec. 4 we briefly discuss our results and we point out some further related problems. We briefly review the construction of the family of commuting transfer matrices for integrable periodic and open quantum spin chains in appendices A and B, respectively.

2 Periodic spin chain

It is useful to begin by briefly reviewing how the compatibility of the auxiliary linear problem for a periodic spin chain (1.3)-(1.4) with (1.5)

$$T_{k+1} Q^{[k]} - T_k^- Q^{[k+2]} = \phi^{[k]} Q^{[-k-2]}, \quad (2.1)$$

$$T_{k-1} Q^{[-k-2]} - T_k^- Q^{[-k]} = -\bar{\phi}^{[-k]} Q^{[k]}, \quad (2.2)$$

implies the Hirota equation (1.1). Multiplying (2.2) by T_{k+1} gives

$$T_{k+1} T_{k-1} Q^{[-k-2]} - T_{k+1} T_k^- Q^{[-k]} = -\bar{\phi}^{[-k]} T_{k+1} Q^{[k]}. \quad (2.3)$$

On the other hand, performing on (2.2) the shifts $k \mapsto k+1$ and $u \mapsto u + \frac{i}{2}$, and then multiplying the result by T_k^- gives

$$T_k^- T_k^+ Q^{[-k-2]} - T_k^- T_{k+1} Q^{[-k]} = -\bar{\phi}^{[-k]} T_k^- Q^{[k+2]}. \quad (2.4)$$

Subtracting (2.3) from (2.4) yields

$$(T_k^+ T_k^- - T_{k+1} T_{k-1}) Q^{[-k-2]} = \bar{\phi}^{[-k]} (T_{k+1} Q^{[k]} - T_k^- Q^{[k+2]}) = \bar{\phi}^{[-k]} \phi^{[k]} Q^{[-k-2]}, \quad (2.5)$$

where the second equality follows from (2.1). It is now clear that (2.5), which expresses the compatibility of (2.1) and (2.2) for the function Q , implies the Hirota equation (1.1). Note also that (2.2) can be obtained from (2.1): performing on (2.1) the shifts $k \mapsto k-1$ and $u \mapsto u + \frac{i}{2}$, we obtain

$$T_k^+ Q^{[k]} - T_{k-1} Q^{[k+2]} = \phi^{[k]} Q^{[-k]}, \quad (2.6)$$

which (up to an overall factor -1) is the complex conjugate of (2.2), assuming that $T_k(u)^* = T_k(u^*)$ is real analytic and $\phi(u)^* = \bar{\phi}(u^*)$. In fact, provided that the last two conditions are met, the entire reasoning can be extended to the general case $Q(u)^* = \bar{Q}(u^*)$.

3 Open spin chain

For an open spin chain, the corresponding function $\phi(u)$ (B.9) does not satisfy the constraint $\bar{\phi} = \phi^{[-2]}$ (1.6) that follows from (1.1). Indeed, the Hirota equation takes a form slightly different from (1.1), namely,

$$T_k^+ T_k^- - T_{k+1} T_{k-1} = T_{2,k}, \quad T_{-1} = 0, \quad k = 0, 1, 2, \dots, \quad (3.1)$$

where $T_{2,k}$ is the quantum determinant

$$T_{2,k} = \prod_{j=0}^{k-1} \phi^{[k-2j]} \bar{\phi}^{[2j-k]}, \quad (3.2)$$

which satisfies the discrete Laplace equation

$$T_{2,k}^+ T_{2,k}^- = T_{2,k+1} T_{2,k-1}. \quad (3.3)$$

Since $T_{2,0} = 1$, Eq. (3.1) with $k = 0$ implies that

$$T_0 = 1, \quad (3.4)$$

which differs from the first relation of (1.6).

3.1 Homogeneous case

For an open spin chain with *diagonal* boundary terms, we propose the following homogeneous auxiliary linear problem

$$T_{k+1} Q^{[k]} - \bar{\phi}^{[k]} T_k^- Q^{[k+2]} = X_k Q^{[-k-2]}, \quad (3.5)$$

$$\phi^{[-k]} T_{k-1} Q^{[-k-2]} - T_k^- Q^{[-k]} = -Y_k Q^{[k]}, \quad (3.6)$$

where

$$X_k = \prod_{j=0}^k \phi^{[k-2j]}, \quad Y_k = \prod_{j=0}^{k-1} \bar{\phi}^{[2j-k]}, \quad (3.7)$$

instead of (2.1)-(2.2). Indeed, following the same steps as in the periodic case (2.3)-(2.5), we find with the help of the simple identities

$$Y_{k+1}^+ = \bar{\phi}^{[k]} Y_k, \quad X_k Y_k = \phi^{[-k]} T_{2,k}, \quad (3.8)$$

that the compatibility of the linear system (3.5)-(3.6) implies the Hirota equation (3.1). Moreover, (3.6) can be obtained from (3.5) in the same way that (2.2) can be obtained from (2.1), see (2.6).

Eq. (3.5) can be solved for T_k in terms of Q . For $k = 0$, one readily obtains the usual T-Q equation

$$T_1 Q = \bar{\phi} Q^{[2]} + \phi Q^{[-2]}, \quad (3.9)$$

as in the periodic case (1.7). The result for general values of k can alternatively be obtained from a generating function [17]

$$\mathcal{W}_{diag} \equiv (1 - \mathcal{D}B\mathcal{D})^{-1} (1 - \mathcal{D}A\mathcal{D})^{-1} = \sum_{k=0}^{\infty} \mathcal{D}^k T_k \mathcal{D}^k, \quad (3.10)$$

where

$$A = \phi \frac{Q^{[-2]}}{Q}, \quad B = \bar{\phi} \frac{Q^{[2]}}{Q}, \quad (3.11)$$

and $\mathcal{D} = e^{-\frac{i}{2}\partial_u}$ implying that $\mathcal{D}f = f^-\mathcal{D}$. In this way, we obtain

$$T_k = \sum_{l=0}^k \prod_{j=0}^{k-l-1} B^{[k-1-2j]} \prod_{i=0}^{l-1} A^{[2l-k-1-2i]}. \quad (3.12)$$

3.2 Inhomogeneous case

For an open spin chain with *non-diagonal* boundary terms, we propose the following inhomogeneous auxiliary linear problem

$$T_{k+1} Q^{[k]} - \bar{\phi}^{[k]} T_k^- Q^{[k+2]} = X_k Q^{[-k-2]} + \sum_{l=0}^k \psi_{l,k} \Delta^{[2l-k]} T_l^{[l-k-1]}, \quad (3.13)$$

$$\phi^{[-k]} T_{k-1} Q^{[-k-2]} - T_k^- Q^{[-k]} = -Y_k Q^{[k]} - \sum_{l=0}^{k-1} \bar{\psi}_{l,k-1}^- \Delta^{[k-2l-2]} T_l^{[k-l-1]}, \quad (3.14)$$

where X_k and Y_k are given by (3.7), and $\psi_{l,k}$ is given by

$$\psi_{l,k} = \prod_{j=0}^{k-l-1} \phi^{[k-2j]}, \quad \bar{\psi}_{l,k} = \prod_{j=0}^{k-l-1} \bar{\phi}^{[2j-k]}. \quad (3.15)$$

For $\Delta = 0$, this system evidently reduces to the homogeneous system (3.5)-(3.6).

Eq. (3.13) can be used to solve for all T_k in terms of Q . The inhomogeneous T-Q equation (1.8) is obtained for $k = 0$. The result for general values of k can again be alternatively obtained from a generating function [17]

$$\mathcal{W} \equiv [1 - \mathcal{D}(A + B + C)\mathcal{D} + \mathcal{D}A\mathcal{D}^2B\mathcal{D}]^{-1} = \sum_{k=0}^{\infty} \mathcal{D}^k T_k \mathcal{D}^k, \quad (3.16)$$

where A and B are again given by (3.11), and C is given by

$$C = \frac{\Delta}{Q}. \quad (3.17)$$

Note that this generating function reduces to \mathcal{W}_{diag} (3.10) for $\Delta = 0$.

3.2.1 Compatibility conditions

We now proceed as in the homogeneous case to derive the compatibility conditions for the auxiliary linear problem (3.13)-(3.14) for the function Q . Multiplying (3.14) by T_{k+1} gives

$$\begin{aligned} T_{k+1} T_{k-1} \phi^{[-k]} Q^{[-k-2]} - T_{k+1} T_k^- Q^{[-k]} &= -Y_k T_{k+1} Q^{[k]} \\ &- \sum_{l=0}^{k-1} \bar{\psi}_{l,k-1}^- \Delta^{[k-2l-2]} T_{k+1} T_l^{[k-l-1]}. \end{aligned} \quad (3.18)$$

On the other hand, performing on (3.14) the shifts $k \mapsto k+1$ and $u \mapsto u + \frac{i}{2}$, and then multiplying the result by T_k^- gives

$$T_k^- T_k^+ \phi^{[-k]} Q^{[-k-2]} - T_k^- T_{k+1} Q^{[-k]} = -Y_k \bar{\phi}^{[k]} T_k^- Q^{[k+2]} - \sum_{l=0}^k \bar{\psi}_{l,k} \Delta^{[k-2l]} T_k^- T_l^{[k-l+1]} \quad (3.19)$$

Subtracting (3.18) from (3.19) yields

$$\begin{aligned} (T_k^+ T_k^- - T_{k+1} T_{k-1}) \phi^{[-k]} Q^{[-k-2]} &= Y_k (T_{k+1} Q^{[k]} - \bar{\phi}^{[k]} T_k^- Q^{[k+2]}) \\ &+ \sum_{l=0}^{k-1} \bar{\psi}_{l,k-1}^- \Delta^{[k-2l-2]} T_{k+1} T_l^{[k-l-1]} - \sum_{l=0}^k \bar{\psi}_{l,k} \Delta^{[k-2l]} T_k^- T_l^{[k-l+1]}. \end{aligned} \quad (3.20)$$

Using (3.13), (3.8) and (3.15), we arrive at

$$\begin{aligned} (T_k^+ T_k^- - T_{k+1} T_{k-1} - T_{2,k}) \phi^{[-k]} Q^{[-k-2]} &= \sum_{l=0}^{k-1} \bar{\psi}_{l,k-1}^- \Delta^{[k-2l-2]} T_{k+1} T_l^{[k-l-1]} - \sum_{l=0}^k \bar{\psi}_{l,k} \Delta^{[k-2l]} T_k^- T_l^{[k-l+1]} + \sum_{l=0}^k \psi_{l,k} Y_k \Delta^{[2l-k]} T_l^{[l-k-1]} \\ &= \sum_{a=0}^{k-1} \Delta^{[-k+2a]} \left(\prod_{j=0}^{a-1} \bar{\phi}^{[-k+2j]} \right) H_{k,a}, \quad k = 1, 2, \dots, \end{aligned} \quad (3.21)$$

where

$$H_{k,a} = T_{k+1} T_{k-a-1}^{[a]} - T_k^- T_{k-a}^{[a+1]} + T_{2,k-a}^{[a]} T_a^{[a-k-1]}. \quad (3.22)$$

Eqs. (3.21) are evidently satisfied for generic Q and Δ if

$$H_{k,a} = 0, \quad a = 0, 1, \dots, k-1, \quad (3.23)$$

which are precisely the Hirota-like bilinear relations (1.12).

3.3 Solving the Hirota equation

It is easy to explicitly solve the open-chain Hirota equation (3.1) for T_k in terms of T_1 for small values of k , and to show that the resulting expressions can be conveniently recast in

terms of determinants

$$\begin{aligned}
T_2 &= \begin{vmatrix} T_1^{[1]} & \phi^{[1]} \\ \bar{\phi}^{[-1]} & T_1^{[-1]} \end{vmatrix}, \\
T_3 &= \begin{vmatrix} T_1^{[2]} & \phi^{[2]} & 0 \\ \bar{\phi}^{[0]} & T_1^{[0]} & \phi^{[0]} \\ 0 & \bar{\phi}^{[-2]} & T_1^{[-2]} \end{vmatrix}, \\
T_4 &= \begin{vmatrix} T_1^{[3]} & \phi^{[3]} & 0 & 0 \\ \bar{\phi}^{[1]} & T_1^{[1]} & \phi^{[1]} & 0 \\ 0 & \bar{\phi}^{[-1]} & T_1^{[-1]} & \phi^{[-1]} \\ 0 & 0 & \bar{\phi}^{[-3]} & T_1^{[-3]} \end{vmatrix}.
\end{aligned} \tag{3.24}$$

This suggests a general determinant expression for T_k in terms of T_1 (see also [9])

$$T_k = \det(M^{(k)}), \tag{3.25}$$

where $M^{(k)}$ is a $k \times k$ matrix whose elements are given by

$$M_{ij}^{(k)} = T_1^{[k+1-2i]} \delta_{ij} + \bar{\phi}^{[k+1-2i]} \delta_{i,j+1} + \phi^{[k+1-2i]} \delta_{i,j-1}, \quad i, j = 1, \dots, k. \tag{3.26}$$

We can now verify that (3.25) is the solution of the Hirota equation using Jacobi's determinant identity [9, 29]

$$D[p_1, p_2 | q_1 q_2] D = D[p_1 | q_1] D[p_2 | q_2] - D[p_1 | q_2] D[p_2 | q_1], \tag{3.27}$$

where D is the determinant of a square matrix, and $D[p_1, p_2, \dots, p_n | q_1, q_2, \dots, q_n]$ denotes the minor determinant obtained from the same matrix by removing rows p_1, p_2, \dots, p_n and columns q_1, q_2, \dots, q_n . Indeed, let us observe that the matrix $M^{(k+1)}$, obtained from (3.26), contains $M^{(k-1)}$ as a submatrix

$$M^{(k+1)} = \begin{pmatrix} T_1^{[k]} & \phi^{[k]} & 0 & \dots & 0 & 0 & 0 \\ \bar{\phi}^{[k-2]} & & & & & & 0 \\ \vdots & & & M^{(k-1)} & & & \vdots \\ 0 & & & & & & \phi^{[-k+2]} \\ 0 & 0 & 0 & \dots & 0 & \bar{\phi}^{[-k]} & T_1^{[-k]} \end{pmatrix}. \tag{3.28}$$

Applying the Jacobi identity (3.27) to the above $(k+1) \times (k+1)$ matrix with $p_1 = q_1 = 1$ and $p_2 = q_2 = k+1$, and then using (3.25), we recover the Hirota equation (1.1). (Note that the matrices corresponding to $D[1|k+1]$ and $D[k+1|1]$ are either upper or lower triangular, with ϕ 's along the diagonal.)

We remark that Eq. (3.25) provides an expression for T_k in terms of Q upon setting $T_1 = A + B + C$ (see (3.11), (3.17)) in (3.26). In particular, for the diagonal case $\Delta = 0$, the result is equivalent to (3.12).

3.4 Solving the Hirota-like equations

We now demonstrate that the solution (3.25) of the Hirota equation is also a solution of the Hirota-like equations (1.12). The main idea is to use Plücker relations [9, 29], which are generalizations of Jacobi's identity. In this way, we see that the Hirota equation stems from the Jacobi identity, while the generalizations of the Hirota equation stem from the extension of the Jacobi identity to the Plücker relations.

Let X be a *rectangular* matrix with $n + 1$ rows and $r + 1$ columns ($n \geq r$),

$$X = \begin{pmatrix} X_{0,0} & X_{0,1} & \cdots & X_{0,r} \\ X_{1,0} & X_{1,1} & \cdots & X_{1,r} \\ \vdots & \vdots & \cdots & \vdots \\ X_{n,0} & X_{n,1} & \cdots & X_{n,r} \end{pmatrix}. \quad (3.29)$$

Following [9], we define (i_0, i_1, \dots, i_r) to be the determinant of the square matrix formed by the rows with labels i_0, i_1, \dots, i_r ,

$$\begin{vmatrix} X_{i_0,0} & X_{i_0,1} & \cdots & X_{i_0,r} \\ X_{i_1,0} & X_{i_1,1} & \cdots & X_{i_1,r} \\ \vdots & \vdots & \cdots & \vdots \\ X_{i_r,0} & X_{i_r,1} & \cdots & X_{i_r,r} \end{vmatrix} \equiv (i_0, i_1, \dots, i_r). \quad (3.30)$$

It is understood that (i_0, i_1, \dots, i_r) is antisymmetric in all the indices, and therefore vanishes if any two indices coincide. The Plücker relations are then given by [9]

$$(i_0, i_1, \dots, i_r)(j_0, j_1, \dots, j_r) = \sum_{p=0}^r (j_p, i_1, \dots, i_r)(j_0, \dots, j_{p-1}, i_0, j_{p+1}, \dots, j_r) \quad (3.31)$$

for all $i_p, j_p \in \{0, n\}$ with $p = 0, 1, \dots, r$.

It is convenient to introduce the notation $[i, p]$ as follows

$$(X_{i,0}, X_{i,1}, \dots, X_{i,r}) \Big|_{[i,p]} = (X_{i,0}, X_{i,1}, \dots, X_{i,r}) \Big|_{X_{i,j}=\delta_{j,p}} = (\overset{0}{\downarrow} 0, \dots, \overset{p}{\downarrow} 1, 0, \dots, \overset{r}{\downarrow} 0). \quad (3.32)$$

In other words, $[i, p]$ means that the row of matrix X labeled i is given by $(0, \dots, 0, 1, 0, \dots, 0)$, where the 1 appears in the column labeled p . Similarly, for a set of m rows of the matrix X , we define

$$\begin{pmatrix} X_{i_1,0} & X_{i_1,1} & \cdots & X_{i_1,r} \\ X_{i_2,0} & X_{i_2,1} & \cdots & X_{i_2,r} \\ \vdots & \vdots & \cdots & \vdots \\ X_{i_m,0} & X_{i_m,1} & \cdots & X_{i_m,r} \end{pmatrix} \Big|_{\begin{bmatrix} i_1, p_1 \\ i_2, p_2 \\ \vdots \\ i_m, p_m \end{bmatrix}} = \begin{pmatrix} X_{i_1,0} & X_{i_1,1} & \cdots & X_{i_1,r} \\ X_{i_2,0} & X_{i_2,1} & \cdots & X_{i_2,r} \\ \vdots & \vdots & \cdots & \vdots \\ X_{i_m,0} & X_{i_m,1} & \cdots & X_{i_m,r} \end{pmatrix} \begin{matrix} X_{i_1,j} = \delta_{j,p_1} \\ X_{i_2,j} = \delta_{j,p_2} \\ \vdots \\ X_{i_m,j} = \delta_{j,p_m} \end{matrix} \quad (3.33)$$

We are now ready to show that the solution of the Hirota equation (1.9) is also solution to the Hirota-like equations (1.12). We set

$$r = k, \quad n = r + a + 2. \quad (3.34)$$

We then choose the matrix X (3.29) such that its first $r + 1$ rows are given by the matrix $M^{(r+1)}$ in (3.26)

$$\begin{pmatrix} X_{0,0} & X_{0,1} & \dots & X_{0,r} \\ X_{1,0} & X_{1,1} & \dots & X_{1,r} \\ \vdots & \vdots & \dots & \vdots \\ X_{r,0} & X_{r,1} & \dots & X_{r,r} \end{pmatrix} = M^{(r+1)}, \quad (3.35)$$

and we choose the remaining rows of X as follows

$$\begin{bmatrix} r+1, & 0 \\ r+2, & r-a \\ r+3, & r-a+1 \\ \vdots & \vdots \\ r+a+2, & r \end{bmatrix}, \quad (3.36)$$

where we have used the notation (3.33). The Plücker relations (3.31) for this matrix with the following choice of indices ²

$$\begin{aligned} i_l &= j_l = l, & l &= 1, 2, \dots, r-a-1, \\ j_0 &= 0, \quad j_l = l, & l &= r-a, r-a+1, \dots, r, \\ i_0 &= r+1, \quad i_l = l+a+2, & l &= r-a, r-a+1, \dots, r, \end{aligned} \quad (3.37)$$

can be shown to coincide (using the identification (3.25)) with the Hirota-like equations (1.12). We conclude that the solution (3.25), (3.26) of the original Hirota equation (1.9) also satisfies the Hirota-like equations (1.12), and therefore the latter system of equations is consistent with it. We remark that we have not succeeded to find a simple transformation that maps the Hirota-like equations (1.12) to the Hirota equations (1.9).

4 Discussion

We have shown that the $sl(2)$ Hirota equation (1.9) admits a Lax representation with inhomogeneous terms (1.10)-(1.11), thereby demonstrating that the off-diagonal Bethe ansatz approach [15, 16] can be accommodated within the conventional framework of quantum integrability. In so doing, we have found a family of Hirota-like equations (1.12) which are consistent with the original Hirota equation. We expect that (1.10)-(1.11) is the most general Lax representation of the Hirota equation with a single function $Q(u)$, and therefore it

²In [9] it is assumed that $i_p = j_p$ for $p \neq 0, 1$ in order to reduce the number of terms to 3. We do not make this assumption here, but the number of terms nevertheless reduces to 3 by virtue of our choice (3.36).

should describe the most general rank-one integrable quantum system. It may be interesting to work out the generalization to higher-rank algebras and superalgebras. It may also be interesting to investigate the analogue of such inhomogeneous terms in conformal field theories and classical integrable systems with boundaries.

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A Periodic chain transfer matrices

We briefly review here the construction of the family of commuting transfer matrices for a periodic XXX ($sl(2)$ -invariant) quantum spin-1/2 chain with length N , whose Hamiltonian is given by

$$H = \frac{1}{4} \sum_{n=1}^N (\vec{\sigma}_n \cdot \vec{\sigma}_{n+1} - \mathbb{I}) , \quad \vec{\sigma}_{N+1} \equiv \vec{\sigma}_1 . \quad (\text{A.1})$$

The fused $(j, \frac{1}{2})$ R -matrices are given by [13]

$$R_{\{a\}b}^{(j, \frac{1}{2})}(u) = \chi_1^{(j)}(u) P_{\{a\}}^+ \prod_{k=1}^{2j} R_{a_k b}^{(\frac{1}{2}, \frac{1}{2})}(u + (k - j - \frac{1}{2})i) P_{\{a\}}^+ , \quad j = \frac{1}{2}, 1, \frac{3}{2}, \dots \quad (\text{A.2})$$

where $R^{(\frac{1}{2}, \frac{1}{2})}$ is the fundamental $sl(2)$ -invariant R -matrix (solution of the Yang-Baxter equation)

$$R^{(\frac{1}{2}, \frac{1}{2})} = u + i\mathcal{P} , \quad (\text{A.3})$$

and \mathcal{P} is the permutation operator. The R -matrices in the product (A.2) are ordered in the order of increasing k . Moreover, $P_{\{a\}}^+$ is the symmetric projector

$$P_{\{a\}}^+ = \frac{1}{(2j)!} \prod_{k=1}^{2j} \left(\sum_{l=1}^k \mathcal{P}_{a_l, a_k} \right) , \quad (\text{A.4})$$

where $\mathcal{P}_{a_k, a_k} \equiv 1$. Finally, $\chi_1^{(j)}(u)$ is the normalization factor

$$\chi_1^{(j)}(u) = \frac{1}{\prod_{k=0}^{2j-2} \left(u + i\left(j - \frac{1}{2} - k\right)\right)}, \quad (\text{A.5})$$

which removes all trivial zeros.

The corresponding transfer matrices are given by

$$t^{(j)}(u) = \text{tr}_{\{a\}} R_{\{a\} b^{[N]}}^{(j, \frac{1}{2})}(u) \cdots R_{\{a\} b^{[1]}}^{(j, \frac{1}{2})}(u). \quad (\text{A.6})$$

They enjoy the important commutativity property

$$[t^{(j)}(u), t^{(k)}(v)] = 0. \quad (\text{A.7})$$

The Hamiltonian (A.1) is related to the fundamental transfer matrix by

$$H = \frac{i}{2} \frac{d}{du} \ln t^{(\frac{1}{2})}(u) \Big|_{u=0} - \frac{N}{2} \mathbb{I}. \quad (\text{A.8})$$

These transfer matrices are related to the ones discussed in the text as follows

$$T_k(u) = t^{(\frac{k}{2})}(u - \frac{i}{2}), \quad k = 1, 2, 3, \dots \quad (\text{A.9})$$

Indeed, these T_k obey the Hirota equation (1.1) with the functions ϕ and $\bar{\phi}$ given by (1.2), which indeed satisfy the constraint in (1.6).

B Open chain transfer matrices

We now review the construction of the family of commuting transfer matrices for an open chain. In addition to R-matrices, we also need K-matrices (solutions of the boundary Yang-Baxter equation) [21, 30]. For the XXX case, the general fundamental solution is given by [22, 23]

$$K^{(\frac{1}{2})}(u; \alpha, \xi_+, \xi_-) = \begin{pmatrix} i\alpha + u & u\xi_+ \\ u\xi_- & i\alpha - u \end{pmatrix}, \quad (\text{B.1})$$

where α, ξ_+, ξ_- are arbitrary boundary parameters. The fused K -matrices are given by [24, 25, 26]

$$\begin{aligned} K_{\{a\}}^{(j)}(u) &= \chi_2^{(j)}(u) P_{\{a\}}^+ \prod_{k=1}^{2j} \left\{ \left[\prod_{l=1}^{k-1} R_{a_l a_k}^{(\frac{1}{2}, \frac{1}{2})}(2u + (k + l - 2j - 1)i) \right] \right. \\ &\quad \left. \times K_{a_k}^{(\frac{1}{2})}(u + (k - j - \frac{1}{2})i) \right\} P_{\{a\}}^+, \quad j = \frac{1}{2}, 1, \frac{3}{2}, \dots \end{aligned} \quad (\text{B.2})$$

where the products of braces $\{\dots\}$ are ordered in the order of increasing k , and the dependence on the boundary parameters has been suppressed. Moreover, $\chi_2^{(j)}(u)$ is the normalization factor

$$\chi_2^{(j)}(u) = \frac{1}{\prod_{k=1}^{4j-3} \left(u + i\left(j - \frac{k}{2}\right)\right)}. \quad (\text{B.3})$$

For simplicity, we consider here the following right and left K-matrices

$$K^{r(j)}(u) = K^{(j)}(u; \alpha, 0, 0), \quad K^{l(j)}(u) = K^{(j)}(-u - i; \beta, \xi, \xi), \quad (\text{B.4})$$

respectively, with α, β, ξ real. In other words, we choose the right K-matrices to be diagonal, and the left K-matrices to be non-diagonal with $\xi_+ = \xi_- = \xi$.

The corresponding open-chain transfer matrices $t^{(j)}(u)$ are given by [21]

$$t^{(j)}(u) = \text{tr}_{\{a\}} K_{\{a\}}^{l(j)}(u) T_{\{a\}}^{(j)}(u) K_{\{a\}}^{r(j)}(u) \hat{T}_{\{a\}}^{(j)}(u), \quad (\text{B.5})$$

where the monodromy matrices are given by products of N fused R -matrices,

$$\begin{aligned} T_{\{a\}}^{(j)}(u) &= R_{\{a\} b^{[N]}}^{(j, \frac{1}{2})}(u) \dots R_{\{a\} b^{[1]}}^{(j, \frac{1}{2})}(u), \\ \hat{T}_{\{a\}}^{(j)}(u) &= R_{\{a\} b^{[1]}}^{(j, \frac{1}{2})}(u) \dots R_{\{a\} b^{[N]}}^{(j, \frac{1}{2})}(u). \end{aligned} \quad (\text{B.6})$$

The corresponding open-chain Hamiltonian, which is obtained from the fundamental transfer matrix, is given by

$$\begin{aligned} H &= \frac{i(-1)^{N+1}}{2\alpha\beta} \frac{d}{du} t^{(\frac{1}{2})}(u) \Big|_{u=0} - N\mathbb{I} \\ &= \sum_{n=1}^{N-1} \vec{\sigma}_n \cdot \vec{\sigma}_{n+1} + \frac{1}{\alpha} \sigma_1^z - \frac{1}{\beta} (\xi \sigma_N^x + \sigma_N^z). \end{aligned} \quad (\text{B.7})$$

The transfer matrices (B.5) also have the important commutativity property (A.7). We define corresponding T_k which are related to (B.5) in the same way as for the closed chain, namely

$$T_k(u) = t^{(\frac{k}{2})}(u - \frac{i}{2}), \quad k = 1, 2, 3, \dots \quad (\text{B.8})$$

When suitably normalized, these transfer matrices obey [24, 25, 26] the open-chain Hirota equation (3.1), (3.2) with

$$\phi(u) = -\frac{1}{u} \left(u + i\left(\alpha - \frac{1}{2}\right)\right) \left(\sqrt{1 + \xi^2} \left(u - \frac{i}{2}\right) - i\beta\right) \left(u + \frac{i}{2}\right)^{2N+1}. \quad (\text{B.9})$$

(As usual, $\bar{\phi}(u)$ is given by the complex conjugate of $\phi(u)$.) As noted in Sec. 3, this function does not satisfy the constraint (1.6) for generic values of the boundary parameters. When $\xi = 0$, both the left and right K-matrices are diagonal, and the transfer matrix has a $U(1)$ symmetry. If $\xi \neq 0$, then this symmetry is broken.

The T-Q equation for the fundamental transfer matrix is given by (1.8) with $\phi(u)$ given by (B.9), and with Δ given by [15, 16]

$$\Delta = -2 \left(1 - \sqrt{1 + \xi^2} \right) \left(u + \frac{i}{2} \right)^{2N+1} \left(u - \frac{i}{2} \right)^{2N+1}, \quad (\text{B.10})$$

which vanishes for $\xi = 0$, i.e. when the model has a $U(1)$ symmetry. Indeed, one can explicitly check [17] that for each eigenvalue of the transfer matrix, there exists a polynomial function $Q(u)$ that satisfies this T-Q equation.

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